# ''Chirped'' Van der Pol oscillator

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A frequency ''chirp'' can destroy the limit cycle of the Van der Pol oscillator. Simple criteria for the preservation of the limit cycle despite the chirp are found analytically and verified numerically in the cases of a weak and strong nonlinearity. [S1063-651X(97)06007-8]

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Dynamics of physical systems, the parameters of which slowly (adiabatically) vary in time, have always attracted attention. The adiabatic invariant [1] that exists in the problem of a (linear or nonlinear) frictionless oscillator, with a time-dependent frequency, is the best known elementary example. More intricate are examples from the nonintegrable Hamiltonian dynamics. We can mention here the problem of an adiabatic invariant in the chaotic systems with slowly time-dependent parameters [2] and the problem of the ''slowly varying chaos'' in the systems where both fast and slow time dependences are present [3].

An open question concerns the dynamics of nonlinear *dissipative* systems with slowly varying parameters. One of the simplest examples of such a system is a natural extension of the famous Van der Pol (VdP) oscillator. The VdP oscillator, introduced in 1920 as a simple model of a vacuum-tube based signal generating circuit [4], is regarded as a standard model of a periodic self-oscillatory (limit-cycle) behavior [5–8]. Examples of systems describable, within an approximation, by the VdP-oscillator model can be found in many fields of physics. For example, they include the classical Froude pendulum [5] (a mechanical pendulum coupled by friction to a rotating shaft, around which it can oscillate). Another well-known example is the tunnel diode generator of periodic signals [7] that employs the ''negative resistance'' property of tunnel diodes [9].

The dynamics of VdP oscillators are controlled by parameters that determine the basic oscillation frequency and degree of nonlinear friction. If these are constant, the limit cycle of a VdP oscillator is unique. In some applications, these parameters can vary in time (for example, the frequency can be ''chirped''). Therefore, a question arises on how these time variations affect the dynamics. Will the limit cycle persist, and if so, under what conditions? In this paper we will answer this question by considering the VdP equation with a time-dependent frequency $\omega(t)$

$$\ddot{x}+\epsilon(x^2-1)\dot{x}+\omega^2(t)x=0\,, \qquad (1)$$

where $\epsilon>0$ is the nonlinearity degree. We will consider separately the cases of a small and large $\epsilon$, extend the corresponding perturbation techniques to the case of a chirped frequency, and compare the results with numerical solutions.

Let us start with the case of a weak nonlinearity $\epsilon \ll \omega(t)$ and introduce a (constant) characteristic frequency $\omega_0$. It is convenient to work with a scaled time $t'=\omega_0 t$, frequency $\omega(t')=\omega(t)/\omega_0$, and nonlinearity degree $\epsilon'=\epsilon/\omega_0$. With the scaled quantities, the VdP equation coincides with Eq. (1) with the coefficient $\epsilon'\ll 1$ instead of $\epsilon$. We will omit the primes and regard $\epsilon$ as a small parameter. We also require the adiabaticity condition $|\omega^{-2}d\omega/dt|\ll 1$ and look for the solution in the form $x(t)=a(t)\cos[\int_0^t\omega(t')dt'+\phi(t)]$, where $\phi(t)$ is a slow varying phase. Employing the standard method of averaging [5–7], we arrive at the following reduced equations:

$$\dot{a}=\frac{\epsilon a}{2}\left(1-\frac{a^2}{4}\right)-\frac{\dot{\omega}a}{2\omega}, \qquad \dot{\phi}=0 \qquad (2)$$

so that $\phi(t)=\phi(0)=\text{const}$. One can see that if $\omega(t)=\text{const}$, Eq. (2) reduces to the well-known amplitude equation for the stable limit cycle of the VdP oscillator [4–7]. On the other hand, in the conservative case, $\epsilon=0$, Eq. (2) describes the preservation of the adiabatic invariant $I=(1/2)\omega(t)a^2$.

It turns out that Eq. (2) is solvable analytically. Indeed, we introduce a new variable $u=I^{-1}=2\omega(t)^{-1}a^{-2}$. Then Eq. (2) becomes linear,

$$\dot{u}+\epsilon u=\frac{\epsilon}{2\omega(t)}, \qquad (3)$$

and can be readily integrated. Returning to the original variable $a(t)$, we obtain

$$a(t)=\left[\frac{2}{\omega(t)}\right]^{1/2}\left[\frac{2\,e^{-\epsilon t}}{\omega(0)a^2(0)}+\frac{\epsilon}{2}e^{-\epsilon t}\int_0^t\frac{e^{\epsilon t'}dt'}{\omega(t')}\right]^{-1/2}, \qquad (4)$$

where $a(0)$ and $\omega(0)$ are the initial values of the amplitude and frequency, respectively.

The result (4) will obviously depend on the specific form of the function $\omega(t)$. However, some simple and general conclusions about the long time behavior of the system can be drawn immediately. They can be conveniently summarized as the following two theorems (sufficient conditions).

*Theorem 1.* For $\epsilon\ll\omega(t)$, if there exists a positive $\alpha$ such that at any time $t>0$

$$\frac{1}{\omega}\frac{d\omega}{dt}<\alpha<\epsilon, \qquad (5)$$

then any trajectory of the dynamic system (2) [except $a(0)=0$] approaches a limit cycle with the amplitude

 

$$a(t\to\infty)>2(1-\alpha/\epsilon)^{1/2}. \tag{6}$$

*Proof.* First, we integrate both sides of the left inequality of Eq. (5) from zero to $t$ and obtain $\omega(t)/\omega(0)<e^{\alpha t}$. We multiply both parts of this inequality by $e^{-\epsilon t}$, take the limit of $t\to\infty$ and, using the right inequality of Eq. (5), obtain

$$\lim_{t\to\infty}\frac{\omega(t)}{\omega(0)}e^{-\epsilon t}=0\,. \tag{7}$$

Next consider the integral

$$\int_0^t\frac{e^{\epsilon t'}dt'}{\omega(t')}=\frac{1}{\epsilon}\left[\frac{e^{\epsilon t'}}{\omega(t)}-\frac{1}{\omega(0)}\right]+\frac{1}{\epsilon}\int_0^t\frac{e^{\epsilon t'}}{\omega^2(t)}\frac{d\omega}{dt'}dt'. \tag{8}$$

Using the left inequality of Eq. (5) in the second term on the right-hand side and rearranging the terms, we arrive at

$$\int_0^t\frac{e^{\epsilon t'}dt'}{\omega(t')}<\frac{1}{\epsilon-\alpha}\left[\frac{e^{\epsilon t}}{\omega(t)}-\frac{1}{\omega(0)}\right]. \tag{9}$$

We use this inequality in Eq. (4) to obtain

$$a(t)>2^{1/2}\left\{\left[\frac{2}{a^2(0)}-\frac{\epsilon}{2(\epsilon-\alpha)}\right]\frac{\omega(t)}{\omega(0)}e^{-\epsilon t}+\frac{\epsilon}{2(\epsilon-\alpha)}\right\}^{-1/2}. \tag{10}$$

Taking the limit of $t\to\infty$ on both sides and using Eq. (7), we obtain the required inequality (6).

*Theorem 2.* For $\epsilon\ll\omega(t)$, if there exists $\alpha$ such that at any time $t>0$

$$\epsilon<\alpha<\frac{1}{\omega}\frac{d\omega}{dt}, \tag{11}$$

then the amplitude $a(t)$ will approach zero (no limit cycle)

$$a(t\to\infty)=0\,. \tag{12}$$

*Proof.* First, it is easy to prove that

$$\lim_{t\to\infty}\frac{\omega(t)}{\omega(0)}e^{-\epsilon t}=\infty. \tag{13}$$

Also, one can show that

$$\int_0^t\frac{e^{\epsilon t'}dt'}{\omega(t')}>\frac{1}{\epsilon-\alpha}\left[\frac{e^{\epsilon t}}{\omega(t)}-\frac{1}{\omega(0)}\right]. \tag{14}$$

Using this inequality in Eq. (4) and going to the limit of $t\to\infty$ one obtains, in view of Eq. (13), the result (12).

In simple words, Theorems 1 and 2 say that if the frequency $\omega(t)$ is varying slowly enough, the (slowly varying) stable limit cycle persists. On the contrary, too rapid a frequency chirp can destroy the limit cycle.

In order to verify the analytical result (4) and predictions of Theorems 1 and 2, we solved Eq. (1) numerically for different values of parameters. Two representative examples of these computations are shown in Fig. 1. The chirped frequency was $\omega(t)=10e^{\alpha t}$. In this case the perturbation theory predicts that, for $\alpha<\epsilon$ the limit cycle persists, while its amplitude $a$ is constant and equal to $2\,(1-\alpha/\epsilon)^{1/2}$. On the contrary, for $\alpha>\epsilon$ the limit cycle is destroyed, and $a(t\to\infty)=0$. We took the nonlinearity degree $\epsilon=1.0$. (We have returned to the original variables. In the scaled variables $\epsilon'=0.1$.) Figure 1(a) corresponds to two different initial conditions for $\alpha=0.5$. It is seen that $a\to\sqrt{2}$ as predicted. Figure 1(b) refers to the case of $\alpha=5.0$. One can see that $a\to 0$ as predicted. In Fig. 1(a) the predicted amplitude dynamics [Eq. (4)] agrees very well with the computed one. The agreement is less good in Fig. 1(b) because the adiabaticity condition $|\omega^{-2}d\omega/dt|\ll 1$ is not satisfied at early times.

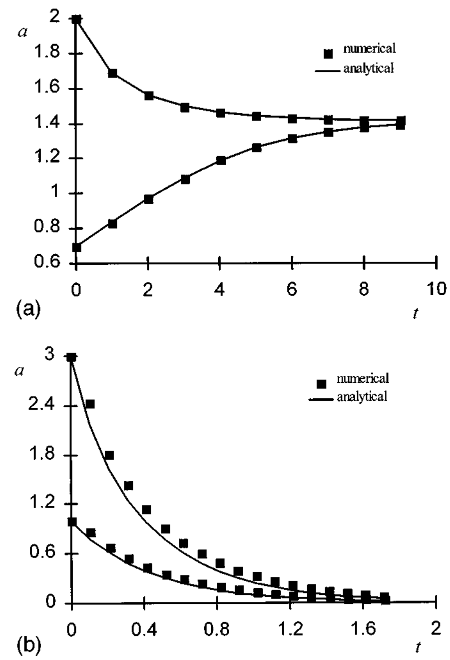


FIG. 1. The VdP-oscillator amplitude vs time as predicted by the analytical expression (4) and numerical solution for different initial conditions. Parameter $\epsilon=1.0$, while $\alpha=0.5$ (a) and 5.0 (b).

Now let us return to the original Eq. (1) and consider the opposite case of a very strong nonlinearity, $\epsilon\gg\omega(t)$. Following the standard technique of the slow and fast motions in relaxation oscillations [5,7], we introduce a slow time $t'=t/\epsilon$ and auxiliary variable $z=x-x^3/3-\epsilon^{-2}dx/dt'$. Now Eq. (1) can be rewritten as a set of two first order equations

$$\frac{dz}{dt'}=\omega^2(\epsilon t')x, \tag{15}$$

$$\epsilon^{-2}\frac{dx}{dt'}=x-\frac{x^3}{3}-z. \tag{16}$$

One can see that the ''fast'' equation (16) coincides with that for the VdP oscillator with a constant frequency. Correspondingly, the phase plane $z,x$ of the system looks similar to the constant frequency case (Fig. 2). For not too large a frequency $\omega(\epsilon t')$, a generic phase point (for example, point $E$ in Fig. 2) moves very fast (almost vertically) towards one of the two stable branches of the slow motion curve $x-x^3/3-z=0$. Then it moves slowly along the curve, as described by Eq. (15), until it reaches one of the extremum points, $(\frac{2}{3},1)$ or $(-\frac{2}{3},-1)$. Here it ''jumps'' almost vertically

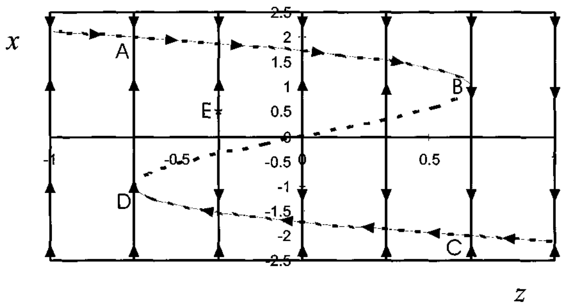


FIG. 2. Phase plane $z,x$ of a highly nonlinear VdP oscillator with a time-dependent frequency.

to the other stable branch of the slow motion curve, the motion along the curve continues and so on. The slow motion equation (15), however, is significantly different from that in the constant frequency case. In particular, it is clear that if $\omega(\epsilon t')$ goes to zero too fast, the motion along the slow curve will stop and, correspondingly, the limit cycle will cease to exist. To check this quantitatively, let us estimate the time $T(n)$ it takes the phase point to complete $n$ cycles. Using Eqs. (15) and (16) we have

$$dt'=\frac{dz}{\omega^2(\epsilon t')x}\simeq\frac{d(x-x^3/3)}{\omega^2(\epsilon t')x}. \tag{17}$$

It is convenient to calculate $\int_0^t\omega^2(\epsilon t')dt'$. For simplicity, we consider the phase point that starts from point $A$ (see Fig. 2), and neglect the time needed to traverse the fast, almost vertical segments of the trajectory. Using Eq. (17) we obtain

$$\int_0^{T(n)}\omega^2(\epsilon t')dt'=n(3-2\ln 2) \tag{18}$$

(we used the fact that $x_A=2$ and $x_B=1$). This equation has a solution $T(n)$ for every $n>0$ if and only if

$$\int_0^{\infty}\omega^2(t)dt=+\infty. \tag{19}$$

Therefore, we have proved the following

*Theorem 3.* For $\epsilon\gg\omega(t)$, the (time-dependent) limit cycle persists for all $t>0$, if and only if criterion (19) is satisfied. Otherwise, $x(t)$ approaches a constant value.

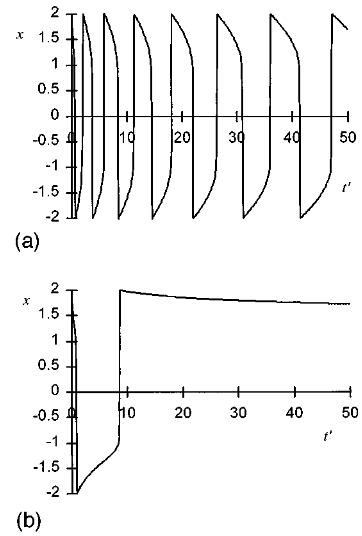


FIG. 3. The dependence $x=x(t')$ found numerically for a highly nonlinear VdP oscillator with a time-dependent frequency. Parameter $\beta=1/4$ (a) and 3/4 (b).

We verified this prediction by numerically solving Eq. (1) for $\epsilon\gg\omega(t)$ and different forms of $\omega(t)$. Figure 3 shows two typical results of such calculations. We chose $\epsilon=10.0$ and started from the same initial conditions $x(0)=2.0$ and $\dot{x}(0)=0$. The time dependence of the frequency was $\omega(\epsilon t')=(1+t')^{-\beta}$, and we varied the parameter $\beta$. Figure 3(a) corresponds to the case of $\beta=1/4$ when criterion (19) is satisfied. One can see that the (time-dependent) limit cycle persists. On the contrary, in the case of $\beta=3/4$, shown in Fig. 3(b), the integral entering criterion (19) is equal to $2.0<+\infty$. One can see that, after a transient, $x(t')$ approaches a constant value, as predicted by Theorem 3.

In summary, we have considered a ''chirped'' VdP oscillator and found, separately in the weakly and strongly nonlinear cases, simple criteria for the persistence of the limit-cycle-type behavior. These criteria and other predictions of the theory agree well with numerical computations. Natural next questions concern the role of slow parameter variations in discrete self-oscillating systems with a higher dimension (like the Lorenz system [10]), and in continuous self-oscillating systems.

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